Bed-Load Transport Equation for Sheet Flow

Athol D. Abrahams[1]

Abstract: When open-channel flows become sufficiently powerful, the mode of bed-load transport changes from saltation to sheet flow. Where there is no suspended sediment, sheet flow consists of a layer of colliding grains whose basal concentration approaches that of the stationary bed. These collisions give rise to a dispersive stress that acts normal to the bed and supports the bed load. An equation for predicting the rate of bed-load transport in sheet flow is developed from an analysis of 55 flume and closed conduit experiments. The equation is $i_b = \omega$ where i_b = immersed bed-load transport rate; and ω = flow power. That $i_b = \omega$ implies that $e_b = \tan\alpha = u_b/u$, where e_b = Bagnold's bed-load transport efficiency; u_b = mean grain velocity in the sheet-flow layer; and $\tan\alpha$ = dynamic internal friction coefficient. Given that $\tan\alpha \approx 0.6$ for natural sand, $u_b \approx 0.6u$, and $e_b \approx 0.6$. This finding is confirmed by an independent analysis of the experimental data. The value of 0.60 for e_b is much larger than the value of 0.12 calculated by Bagnold, indicating that sheet flow is a much more efficient mode of bed-load transport than previously thought.

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Introduction

When open-channel flows transporting noncohesive sediments become sufficiently powerful, the mode of bed-load transport changes from saltation to sheet flow (Bagnold 1966; Sumer et al. 1996). Where there is no suspended sediment, sheet flow consists of a layer of colliding grains whose basal concentration approaches that of the stationary bed. As these grains are accelerated by the flow, they extract momentum from the faster moving fluid and transfer it to the bed via intergranular (i.e., grain-to-grain and grain-to-bed) collisions. These collisions give rise to a dispersive stress that acts normal to the bed and supports the bed load. This type of sheet flow is termed "no-suspension" or "collisional" sheet flow (Sumer et al. 1996; Jenkins and Hanes 1998) and is the focus of this study.

Since the seminal work of Bagnold (1954, 1956, 1966), considerable progress has been made in characterizing and understanding no-suspension sheet flow (e.g., Wilson 1966; Bailard and Inman 1979; Hanes and Bowen 1985; Hanes and Inman 1985a,b; Wilson 1987, 1989; Nnadi and Wilson 1992, 1995; Asano 1995; Sumer et al. 1996; Jenkins and Hanes 1998; Pugh and Wilson 1999), and several bed-load transport models have been developed specifically for this mode of transport (e.g., Wilson 1966; Engelund 1981; Hanes and Bowen 1985; Nnadi and Wilson 1992; Jenkins and Hanes 1998). During the past few decades these models have become increasingly complex as investigators have sought to represent with greater realism the mechanics of the transport process. In contrast, the bed-load transport equation proposed here is extremely simple and entirely empirical. Its form is suggested by Williams' (1970) experiments, and its validity is confirmed by comparing it with data taken from the literature.

Williams (1970) conducted a large set of experiments in flumes of different widths using a well-sorted sand with a median diameter of 1.35 mm. Although Williams characterized the bed as "flat" in 26 of these experiments, the dimensionless shear stress $\theta = \tau/[g(\rho_s - \rho)D$ never exceeds 0.67, where $\tau = \rho g r_b S$ is the bed shear stress, g is the acceleration of gravity, ρ_s = grain density; ρ = fluid density; r_b = hydraulic radius due to the bed; D = median grain diameter; and S = slope of the hydraulic grade line. Fig. 1 displays a graph of the immersed bed-load transport

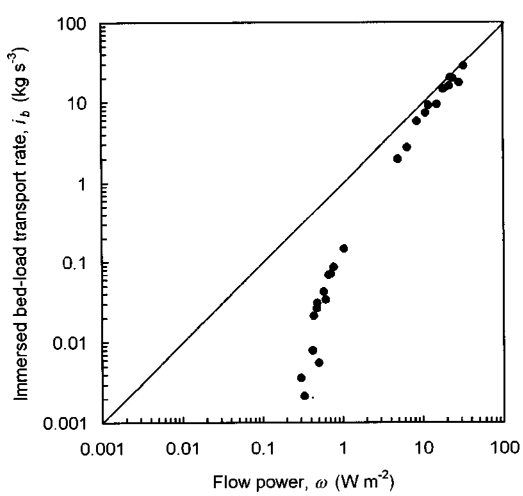

Fig. 1. Graph of immersed bed-load transport rate against flow power for 26 flat bed experiments by Williams (1970). Flow power was calculated using data for hydraulic radius due to bed provided by Williams (1970, Table 2).

[1]Distinguished Professor, Dept. of Geography, Univ. at Buffalo, State Univ. of New York, Buffalo, NY 14261.

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rate $i_b = q_b g(\rho_s - \rho)$ for these 26 experiments against flow power $\omega = \tau u$, where q_b = volumetric bed-load transport rate; and u = mean flow velocity. The plotted points display a curvilinear trend that converges on the relation

$$i_b = \omega \tag{1}$$

It is hypothesized on the basis of Fig. 1 that bed-load transport rates in sheet flow on upper regime plane beds conform to Eq. (1). This hypothesis is tested by comparing the predictions of Eq. (1) with measured total-load transport rates (which equal bed-load transport rates) for no-suspension sheet flows reported in the literature.

Data

Published data on rates of bed-load transport in sheet flow are scarce. Consequently, the data employed in this study come from just three sources, namely, Smart (1984), Rickenmann (1991), and Nnadi and Wilson (1992).

Smart (1984) reported data for 40 flume experiments in which well-sorted gravels with median diameters of 4.2 and 10.5 mm were transported over plane beds. Measurements of hydraulic radius and water discharge were corrected for the effects of flume wall drag using the procedure of Einstein (1950). Bed slopes β ranged from 1.7° (3%) to 11.3° (20%). Consequently, they were sufficiently steep for the downslope component of gravity to contribute significantly to bed shear stress and, hence, to bed-load transport. The downslope component of gravity was incorporated into the slope variable S by dividing $\sin\beta$—the conventional slope variable—by the factor $\sin(\phi_r - \beta)/\sin\phi_r$ (Schoklitsch 1914; see van Rijn 1993, pp. 4.11 and 7.29), where ϕ_r is the residual angle of repose for the grains in transport, which is herein assumed to be 32° (63%).

Rickenmann (1991) performed a similar set of 50 flume experiments in which the transport rate of bed sediments with a mean diameter of 10 mm was measured in flows containing suspended clay with concentrations up to 0.17 and densities ranging from 998 to 1,286 $\text{kg}\cdot\text{m}^{-3}$. Measurements of hydraulic radius and discharge were corrected for wall drag in the same manner as used by Smart (see above). Because bed slope varied between 4° (7%) and 11.3° (20%), it was necessary to incorporate the downslope component of gravity into S. Again, ϕ_r is assumed to be 32° (63%).

To preserve a plane bed at high shear stresses when antidunes would normally have formed under a free-surface flow, Nnadi and Wilson (1992) conducted 105 experiments in a closed conduit. Hydraulic gradients were created by pressure differences while the test section in the conduit remained horizontal. Consequently, although hydraulic gradients reached 11.6° (21%), there was no need to correct S for the effect of the downslope component of gravity. However, it was necessary to correct the hydraulic radius for wall drag, and this was done by the method developed by Wilson (1965). Experiments were performed on sand ($\rho_s/\rho = 2.67$), Bakelite ($\rho_s/\rho = 1.56$), and nylon ($\rho_s/\rho = 1.14$) grains ranging in size from 0.67 to 3.94 mm.

Test

Smart, Rickenmann, and Nnadi and Wilson completed 195 experiments. However, most of these experiments are unsuitable for testing Eq. (1) because either sheet flow did not occur or the measured transport rate (i.e., the immersed total-load transport rate i_t) included suspended solids. To ensure that only experiments with sheet flow and without suspended solids were included in the analysis, all experiments with either $\theta \leqslant 0.6$ or $w/u_* \leqslant 0.8$ were discarded, where $w = (g\Delta D)^{0.5}$ = inertial settling velocity; $u_* = (g r_b S)^{0.5}$ = shear velocity; and $\Delta = (\rho_s - \rho)/\rho$ = relative grain density.

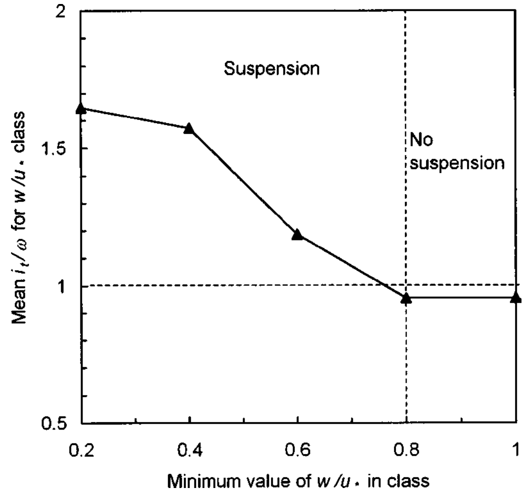

Fig. 2. Graph of mean value of i_t/ω for each w/u_* class against minimum value of w/u_* for each class. Graph is based on data from 155 experiments with $\theta > 0.6$.

The selection of $w/u_* > 0.8$ as the criterion for no suspension is based on Fig. 2, which is a graph of the mean value of i_t/ω against the minimum value of w/u_* in each of the following classes: $1.2 \geqslant w/u_* > 1$, $1 \geqslant w/u_* > 0.8$, $0.8 \geqslant w/u_* > 0.6$, $0.6 \geqslant w/u_* > 0.4$, $0.4 \geqslant w/u_* > 0.2$. The graph shows that, when $w/u_* > 0.8$, i_t/ω is close to 1; but when w/u_* falls below 0.8, the mean value of i_t/ω begins to increase. The logical explanation for this increase is that it reflects the onset of suspension. The selection of w/u_* as the criterion for suspension is consistent with Sumer et al.'s (1996) finding that suspension generally begins in sheet flow when $0.8 < w/u_* < 1$.

The value of θ at which sheet flow commences has been the subject of considerable debate. Sheet flow is generally associated with upper-regime plane beds, but on many so-called plane beds there are low-amplitude bed forms variously termed washed out dunes and ripples (e.g., Wang and White 1993) or low-relief bed waves (e.g., Bennett et al. 1998). Nnadi and Wilson (1995, 1997) found that these bed forms may persist until $\theta = 1$. They therefore argued that $\theta = 1$ represents the critical value above which upper-regime plane beds and sheet flow always exist. Nnadi and Wilson (1995) noted, however, that sheet flow can occur at θ values as low as 0.8, and Sumer et al. (1996) observed sheet flow at θ values as low as 0.6. Although caution dictates utilizing $\theta > 1$ as the criterion for sheet flow (to ensure that experiments with low-amplitude bed forms are excluded), Fig. 3 reveals that the data group closely around the $i_b = \omega$ relation regardless of whether $\theta > 0.6$ or $\theta > 1$ is used as the criterion for inclusion. Given this situation, $\theta > 0.6$ was selected as the criterion for sheet flow in order to maximize the sample size and, hence, the range of experimental conditions.

Fifty-five experiments meet the criteria $\theta > 0.6$ and $w/u_* > 0.8$ for sheet flow with no suspension. The basic data for these experiments are located on the worldwide web at http://geog.buffalo.edu/geomorphology. These experiments represent a

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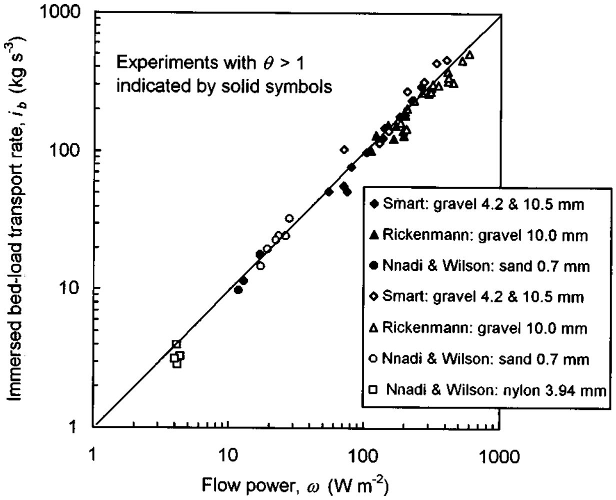

Fig. 3. Graph of immersed bed-load transport rate against flow power for experiments with sheet flow ($\theta > 0.6$) and no suspension ($w/u_* > 0.8$)

diverse mix of experimental methods, grain properties, and flow conditions. The fact that they plot closely around the $i_b = \omega$ relation in Fig. 3 therefore strongly supports the proposition that bed-load transport in sheet flow conforms to this relation.

Alternative Forms

Insofar as Eq. (1) contains only two variables and no coefficients, it is the ultimate parsimonious bed-load transport equation. The two variables, however, are composites of several elementary variables. Eq. (1) can be rewritten in terms of these elementary variables by dividing both sides of the equation by $g(\rho_s - \rho)$. The result is

$$q_b = \frac{r_b u S}{\Delta} = \frac{qS}{\Delta} \tag{2}$$

where q = discharge per unit width. Values of q_b predicted by Eq. (2) are graphed against measured values of q_b in Fig. 4. Clearly

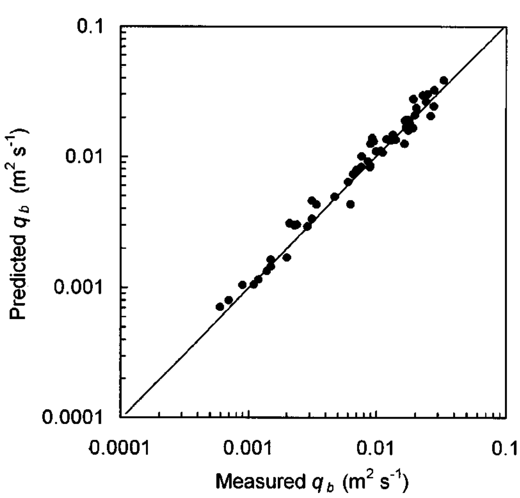

Fig. 4. Graph of q_b predicted by Eq. (2) against measured q_b

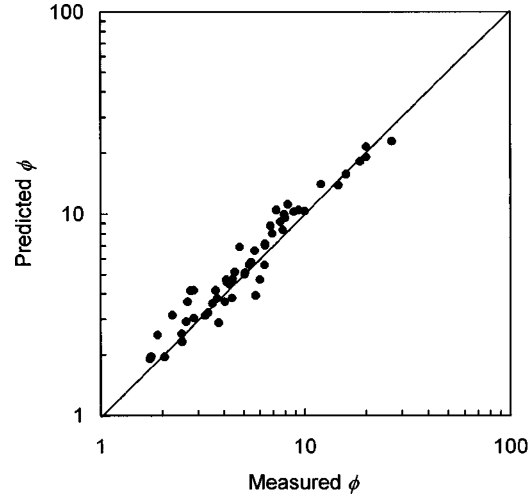

Fig. 5. Graph of ϕ predicted by Eq. (4) against measured ϕ

this equation is a good predictor of the bed-load transport rate in sheet flow.

An intriguing property of Eq. (2) is that it does not contain D. This observation highlights an important difference between saltation and sheet flow. In both modes of transport q_b is dependent on the thickness of the bed-load layer δ. Equations for saltation invariably show δ as a linear function of D (e.g., Bridge and Bennett 1992, Table 6) because the ability of the lift and drag forces to entrain grains is a function of grain size. By contrast, Wilson (1987) suggested that in sheet flow

$$\delta = \frac{\tau}{(\rho_s - \rho) g C \tan \alpha} \tag{3}$$

where C = mean volumetric concentration of solids in the sheet-flow layer. Eq. (3) indicates that δ is linearly dependent on the tangential stress τ and independent of D. This comes about because the entire bed is moving and the immersed weight of the bedload $(\rho_s - \rho) g C \delta$ is equal to the dispersive normal stress regardless of grain size.

Eq. (1) may also be written in the form

$$\phi = \theta^{1.5} \frac{u}{u_*} \tag{4}$$

where $\phi = q_b / [(g \Delta D)^{0.5} D]$ is a dimensionless measure of the bed-load transport rate (Einstein 1950). Eq. (4) is of interest because a number of equations for predicting the bed-load transport rate in sheet flow (e.g., Wilson 1966; Engelund 1981; Jenkins and Hanes 1998) have the form

$$\phi = a \theta^{1.5} \tag{5}$$

where the coefficient a symbolizes a constant or a function that varies from one model to another. Inasmuch as Eq. (4) is algebraically equivalent to Eq. (1), it is expected to be a good predictor of ϕ. This is confirmed by Fig. 5, where the data plot symmetrically around the line of perfect agreement.

Bagnold's Model

The finding that $i_b = \omega$ has important implications for Bagnold's (1966) bed-load transport model as it applies to sheet flow. Bagnold deduced from simple physical principles a quantitative model relating the rate of bed-load transport to the flow properties

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of the transporting fluid. This model is based on the principle that a flow transporting bed load is a machine whose performance may be described by the basic energy equation

$$\text{work rate} = \text{available power} \times \text{efficiency}$$

which he represented by

$$i_b \tan\alpha = \omega \times \frac{u_b}{u} \tag{6}$$

where $\tan\alpha$ = dynamic coefficient of internal friction; and u_b = mean grain velocity. Eq. (6) may be rewritten as

$$i_b = \frac{e_b \omega}{\tan\alpha} \tag{7}$$

where $e_b = u_b/u$ is the efficiency of the flow transporting the bed load. Eq. (7) is Bagnold's (1966) basic bed-load transport equation. The connection between this equation and Eq. (1) is now apparent: if $i_b = \omega$, Eq. (7) implies that

$$e_b = \tan\alpha = \frac{u_b}{u} \tag{8}$$

All three terms in Eq. (8) are important parameters in Bagnold's bed-load transport equation, but none is easy to measure. Of the three parameters, perhaps most effort has been devoted to evaluating $\tan\alpha$. Nevertheless, it is difficult to estimate $\tan\alpha$ for the gravel and nylon grains in the present data set. However, there seems to be a consensus that under inertial conditions, such as occur in sheet flow, $\tan\alpha \approx 0.60$ for natural sand (e.g., Bagnold 1973; Allen and Leeder 1980; Hanes and Bowen 1985; Hanes and Inman 1985a,b; Bridge and Bennett 1992). Given this value for $\tan\alpha$, it follows from Eq. (8) that $u_b/u = e_b \approx 0.6$. Independent support for this value for $u_b/u = e_b$ is provided by the following analysis.

Pugh and Wilson (1999) found that, at the top of the sheet-flow layer, flow velocity

$$u_\delta \approx 9.4 u_* \tag{9}$$

and the velocity profiles in this layer are approximately linear with slopes close to 0.6. From the latter finding it can be shown that

$$u'/u_\delta \approx 0.7 \tag{10}$$

where u' = mean flow velocity in the sheet-flow layer. Substituting Eq. (9) into Eq. (10) and assuming slip velocities are negligible, one obtains

$$u_b \approx u' \approx 6.58 u_* \tag{11}$$

In rough turbulent flow with no side-wall drag (Keulegan 1938, p. 717)

$$\frac{u}{u_*} = 5.75 \log\left(\frac{11 r_b}{k_s}\right) \tag{12}$$

where k_s = equivalent sand roughness, which in sheet flow is given by $k_s \approx \delta/3$ (Pugh and Wilson 1999). Combining Eqs. (11) and (12) produces

$$\frac{u_b}{u} = \frac{u_b}{u_*}\frac{u_*}{u} = \frac{6.58}{5.75 \log(33 r_b/\delta)} \tag{13}$$

Assuming $\Delta = 1.65$, $g = 9.81$, $\tan\alpha = 0.60$, and $C = 0.32$ (i.e., half the loose-poured bed concentration) (Pugh and Wilson 1999), Eq. (3) is reduced to the simpler expression $\delta = 0.32 u_*^2$. Substituting this expression into Eq. (13) yields

$$\frac{u_b}{u} = \frac{1.14}{\log(10.6/S)} \tag{14}$$

Values of u_b/u calculated using Eq. (14) for the 55 sheet-flow experiments range from 0.39 to 0.73 and have a mean and standard deviation of 0.61 and 0.24, respectively. This result corroborates the earlier finding that $e_b = u_b/u \approx 0.60$.

In conclusion, the foregoing discussion and analysis leave little doubt that $e_b \approx 0.60$. This value for e_b is much larger than the value of 0.12 calculated by Bagnold (1966), indicating that sheet flow is a much more efficient mode of transport than previously thought.

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Notation

The following symbols are used in this paper:

- a = coefficient;
- C = mean volumetric concentration of solids in sheet-flow layer;
- D = median grain diameter;
- e_b = bed-load transport efficiency;
- g = acceleration of gravity;
- i_b = immersed bed-load transport rate;
- i_t = immersed total-load transport rate;
- k_s = equivalent sand roughness;
- q = discharge per unit width;
- q_b = volumetric bed-load transport rate;
- r_b = hydraulic radius due to bed;
- S = slope of hydraulic grade line;
- u = mean flow velocity;
- u_b = mean grain velocity in sheet-flow layer;
- u_δ = flow velocity at top of sheet-flow layer;
- u' = mean flow velocity in sheet-flow layer;
- u_* = shear velocity;
- w = settling velocity;
- α = dynamic internal friction angle;
- β = bed slope;
- Δ = relative grain density;
- δ = thickness of sheet-flow layer;
- θ = dimensionless shear stress;
- ρ = fluid density;
- ρ_s = grain density;
- τ = shear stress on bed;
- ϕ = dimensionless bed-load transport rate;
- ϕ_r = residual angle of repose; and
- ω = flow power.

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